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Eigenvalue estimates for the basic Dirac operator on a Riemannian foliation admitting a basic harmonic 1-form

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Abstract

On a compact Riemannian manifold M with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$, if M admits a non-trivial basic harmonic 1-form ω, then any eigenvalue λ of the basic Dirac operator satisfies the inequality $\lambda^2 \geq \frac{q-1}{4(q-2)} \inf_M(\sigma^\nabla + |\kappa|^2)$, where σ^∇ is the transversal scalar curvature and κ is the mean curvature form of $\mathcal{F}$. In the limiting case, $\mathcal{F}$ is minimal and ω is parallel.

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1. Introduction

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. In 2001, Jung [5] proved that any eigenvalue λ of the basic Dirac operator D_b satisfies the inequality

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M K^\sigma, \tag{1.1}$$

where $K^\sigma = \sigma^\nabla + |\kappa|^2$, σ^∇ is the transversal scalar curvature and κ is the mean curvature form of $\mathcal{F}$. In the limiting case, the foliation is minimal, transversally Einsteinian with constant transversal scalar curvature and there is no non-trivial basic harmonic 1-form. This means that on a transverse spin foliation admitting a non-trivial basic harmonic 1-form, there exists a sharper estimate than (1.1). Namely,

Theorem 1.1. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and a bundle-like metric g_M such that $\Delta_B \kappa = 0$. Assume that $K^\sigma > 0$. If M admits a non-trivial basic harmonic 1-form ω of constant length, then any eigenvalue λ of the basic Dirac operator D_b satisfies*

$$\lambda^2 \geq \frac{q-1}{4(q-2)} \inf_M K^\sigma. \tag{1.2}$$

In the limiting case, $\mathcal{F}$ is minimal and ω is parallel.

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In the case of foliations by points, Theorem 1.1 is given by Moroianu and Ornea [10]. Recently, Habib [4] proved lower bounds for the eigenvalues of the basic Dirac operator on a Kähler spin foliation.

2. The generalized Weitzenböck formula

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$ (see [3,5,9] for foliation and [1,2,7] for spin case). We recall the exact sequence

$$0 \to L \to TM \overset{\pi}{\to} Q \to 0 \tag{2.1}$$

determined by the tangent bundle L and the normal bundle $Q = TM/L$ of $\mathcal{F}$. The assumption of g_M being a bundle-like metric means that the induced metric g_Q on the normal bundle $Q \equiv L^\perp$ satisfies the holonomy invariance condition $\theta(X)g_Q = 0$ for all $X \in \Gamma L$, where $\theta(X)$ denotes the Lie derivative with respect to X. The transversal Levi–Civita connection ∇ is given by

$$\nabla_X s = \begin{cases} \pi([X, Y_s]) & \forall X \in \Gamma L \\ \pi(\nabla^M_X Y_s) & \forall X \in \Gamma L^\perp, \end{cases} \tag{2.2}$$

where $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $L^\perp \cong Q$. Let R^∇, ρ^∇ and σ^∇ be respectively the curvature tensor, the transversal Ricci operator and the transversal scalar curvature of $\mathcal{F}$. The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if

$$\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot id \tag{2.3}$$

with constant transversal scalar curvature σ^∇. The *mean curvature form* κ for L is given by

$$\kappa(X) = g_Q\left(\sum_{i=1}^{p} \pi(\nabla^M_{E_i} E_i), X\right) \quad \forall X \in \Gamma Q, \tag{2.4}$$

where $\{E_i\}_{i=1,\dots,p}$ is a local orthonormal basis of L. Let $\Omega^r_B(\mathcal{F})$ be the space of all *basic r-forms*, i.e., $\phi \in \Omega^r_B(\mathcal{F})$ if and only if $i(X)\phi = 0$ and $\theta(X)\phi = 0$ for any $X \in \Gamma L$, where $i(X)$ is an interior product. $\mathcal{F}$ is said to be *minimal* (resp. *isoparametric*) if $\kappa = 0$ (resp. $\kappa \in \Omega^1_B(\mathcal{F})$). It is well-known [11] that if $\mathcal{F}$ is isoparametric on a compact manifold, κ is closed, i.e., $d\kappa = 0$. The *basic Laplacian* Δ_B acting on $\Omega^*_B(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B, \tag{2.5}$$

where δ_B is the formal adjoint of $d_B = d|_{\Omega^*_B(\mathcal{F})}$, which are locally given [5] by

$$d_B = \sum_a E^a \wedge \nabla_{E_a}, \qquad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa^\sharp), \tag{2.6}$$

where $\kappa^\sharp$ is the g_Q-dual vector of κ, $\{E_a\}_{a=1,\dots,q}$ is a local orthonormal basic frame on Q and E^a is its g_Q-dual 1-form. Then we have the following generalized Weitzenböck formula.

Theorem 2.1. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Then the generalized Weitzenböck formula is given by the following: for any basic form $\phi \in \Omega^r_B(\mathcal{F})$*

$$\Delta_B\phi = \nabla^*_{\rm tr}\nabla_{\rm tr}\phi + A_{\kappa^\sharp}(\phi) + F(\phi), \tag{2.7}$$

where

$$\nabla^*_{\rm tr}\nabla_{\rm tr}\phi = -\sum_a \nabla_{E_a}\nabla_{E_a}\phi + \nabla_{\kappa^\sharp}\phi, \tag{2.8}$$

$$A_{\kappa^\sharp}(\phi) = \theta(\kappa^\sharp)\phi - \nabla_{\kappa^\sharp}\phi, \tag{2.9}$$

$$F(\phi) = \sum_{a,b} E^a \wedge i(E_b)R^\nabla(E_b, E_a)\phi. \tag{2.10}$$

In particular, if ϕ is a basic 1-form, then $F(\phi) = \rho^\nabla(\phi^\sharp)$.

Proof. Fix $x \in M$. Let $\{E_a\}$ be a local orthonormal basic frame for Q satisfying $(\nabla E_a)_x = 0$ and $\{E^a\}$ its g_Q-dual basis. Then for any basic form $\phi \in \Omega_B^r(\mathcal{F})$, (2.6) implies

$$d_B\delta_B\phi = -\sum_{a,b} E^a \wedge i(E_b)\nabla_{E_a}\nabla_{E_b}\phi + d_B i(\kappa^\sharp)\phi,$$

$$\delta_B d_B\phi = -\sum_a \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b} E^a \wedge i(E_b)\nabla_{E_b}\nabla_{E_a}\phi + i(\kappa^\sharp)d_B\phi.$$

Summing the above two equations, we have

$$\Delta_B\phi = \theta(\kappa^\sharp)\phi - \sum_a \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b} E^a \wedge i(E_b)R^\nabla(E_b, E_a)\phi,$$

which proves (2.7). On the other hand, let ϕ be a basic 1-form and $\phi^\sharp$ its g_Q-dual vector. Then we have

$$g_Q(F(\phi), E^c) = \sum_b g_Q(R^\nabla(\phi^\sharp, E_b)E_b, E_c) = g_Q(\rho^\nabla(\phi^\sharp), E_c).$$

This yields that for any basic 1-form ϕ, $F(\phi) = \rho^\nabla(\phi^\sharp)$. □

It is well-known [5] that the operator $\nabla_{\text{tr}}^*\nabla_{\text{tr}}$ is non-negative and formally self-adjoint, that is,

$$\int_M g_Q(\nabla_{\text{tr}}^*\nabla_{\text{tr}}\phi, \psi) = \int_M g_Q(\nabla_{\text{tr}}\phi, \nabla_{\text{tr}}\psi) \tag{2.11}$$

for all $\phi, \psi \in \Omega_B^r(\mathcal{F})$, where $g_Q(\nabla_{\text{tr}}\phi, \nabla_{\text{tr}}\psi) = \sum_{a=1}^q g_Q(\nabla_{E_a}\phi, \nabla_{E_a}\psi)$.

Corollary 2.2. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Then for any basic harmonic r-form $\phi \in \Omega_B^r(\mathcal{F})$*

$$-\frac{1}{2}\Delta_B|\phi|^2 = |\nabla_{\text{tr}}\phi|^2 + g_Q(A_{\kappa^\sharp}(\phi), \phi) + g_Q(F(\phi), \phi). \tag{2.12}$$

3. Eigenvalue estimate for basic Dirac operator

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold M with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M such that $\Delta_B\kappa = 0$. The existence of such a metric is assured from [8,9]. Let $S(\mathcal{F})$ be a foliated spinor bundle on a transverse spin foliation $\mathcal{F}$ and $\langle\cdot,\cdot\rangle$ a hermitian scalar product on $S(\mathcal{F})$. By the Clifford multiplication "$\cdot$" in the fibers of $S(\mathcal{F})$ for any vector field $X \in Q$ and any spinor field $\Psi \in S(\mathcal{F})$, the Clifford product $X \cdot \Psi \in S(\mathcal{F})$ is well-defined. This product has the following properties: for any $X, Y \in \Gamma Q$, $Z \in \Gamma TM$ and $\Phi, \Psi \in \Gamma S(\mathcal{F})$,

$$(X \cdot Y + Y \cdot X)\Psi = -2g_Q(X, Y)\Psi \tag{3.1}$$

$$\langle X \cdot \Psi, \Phi\rangle + \langle \Psi, X \cdot \Phi\rangle = 0 \tag{3.2}$$

$$\nabla_Z^S(X \cdot \Psi) = (\nabla_Z X) \cdot \Psi + X \cdot (\nabla_Z^S \Psi), \tag{3.3}$$

where ∇^S is a metric covariant derivation on $S(\mathcal{F})$, i.e., for any $X \in \Gamma TM$ and $\Psi, \Phi \in \Gamma S(\mathcal{F})$, it holds that

$$X\langle \Psi, \Phi\rangle = \langle \nabla_X^S \Psi, \Phi\rangle + \langle \Psi, \nabla_X^S \Phi\rangle. \tag{3.4}$$

From now on, we write ∇ instead of ∇^S. Moreover if we define the Clifford product $\xi \cdot \Psi$ of a 1-form $\xi \in Q^*$ and a spinor field Ψ as

$$\xi \cdot \Psi \equiv \xi^\sharp \cdot \Psi, \tag{3.5}$$

where $\xi^\sharp \in Q$ is a g_Q-dual vector of ξ, then any basic r-form can be considered as an endomorphism of $S(\mathcal{F})$. The curvature transform R^S on $S(\mathcal{F})$ is given [5,7] as

$$R^S(X,Y)\Psi = \frac{1}{4}\sum_{a,b} g_Q(R^\nabla(X,Y)E_a, E_b)E_a\cdot E_b\cdot\Psi, \quad \forall X,Y\in \Gamma TM. \tag{3.6}$$

Then we have the following proposition.

Proposition 3.1 ([5]). *On the transverse spin foliation $\mathcal{F}$, we have the following: for any spinor $\Psi\in\Gamma S(\mathcal{F})$*

$$\sum_{a<b} E_a\cdot E_b\cdot R^S(E_a,E_b)\Psi = \frac{1}{4}\sigma^\nabla\Psi, \tag{3.7}$$

$$\sum_a E_a\cdot R^S(X,E_a)\Psi = -\frac{1}{2}\rho^\nabla(X)\cdot\Psi, \quad \forall X\in\Gamma Q. \tag{3.8}$$

The transversal Dirac operator $D_{\rm tr}$ acting on sections of $S(\mathcal{F})$ is locally given by [3,5,6]

$$D_{\rm tr}\Psi = \sum_a E_a\cdot\nabla_{E_a}\Psi - \frac{1}{2}\kappa\cdot\Psi, \tag{3.9}$$

where $\{E_a\}_{a=1,\dots,q}$ is a local orthonormal basic frame in Q.

Proposition 3.2 ([3,5]). *On an isoparametric transverse spin foliation $\mathcal{F}$ with $\delta_B\kappa = 0$, the Lichnerowicz type formula is given by*

$$D_{\rm tr}^2\Psi = \nabla_{\rm tr}^*\nabla_{\rm tr}\Psi + \frac{1}{4}K^\sigma\Psi, \tag{3.10}$$

where $K^\sigma = \sigma^\nabla + |\kappa|^2$.

At any point $x\in M$, we choose normal coordinates at this point so that $(\nabla E_a)(x) = 0$ for all a. From now on, all the computations in this paper will be made in such charts.

Lemma 3.3. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M. Then for any basic harmonic 1-form $\omega\in\Omega_B^1(\mathcal{F})$*

$$D_{\rm tr}(\omega\cdot\Psi) = -\omega\cdot D_{\rm tr}\Psi - 2\nabla_{\omega^\sharp}\Psi. \tag{3.11}$$

Proof. For any spinor Ψ, a simple calculation gives

$$\begin{aligned} D_{\rm tr}(\omega\cdot\Psi) &= -\sum_a \omega\cdot E_a\cdot\nabla_{E_a}\Psi - 2\nabla_{\omega^\sharp}\Psi + \frac{1}{2}\omega\cdot\kappa\cdot\Psi + (d_B\omega + \delta_B\omega - i(\kappa^\sharp)\omega)\Psi + g_Q(\kappa,\omega)\Psi \\ &= -\omega\cdot D_{\rm tr}\Psi - 2\nabla_{\omega^\sharp}\Psi + (d_B\omega+\delta_B\omega)\Psi. \end{aligned}$$

Since ω is a basic harmonic 1-form, $d_B\omega = \delta_B\omega = 0$. Hence the proof is completed. □

Now, we define the subspace $\Gamma_B S(\mathcal{F})$ of *basic* or *holonomy invariant* sections of $S(\mathcal{F})$ by

$$\Gamma_B S(\mathcal{F}) = \{\Psi\in\Gamma S(\mathcal{F}) | \nabla_X\Psi = 0, \forall X\in\Gamma L\}.$$

Then we see that $D_{\rm tr}$ leaves $\Gamma_B S(\mathcal{F})$ invariant if and only if $\kappa\in\Omega_B^1(\mathcal{F})$. Let $D_b = D_{\rm tr}|_{\Gamma_B S(\mathcal{F})} : \Gamma_B S(\mathcal{F}) \to \Gamma_B S(\mathcal{F})$. This operator D_b is called the *basic Dirac operator* on (smooth) basic sections. It is well-known [3] that D_b and D_b^2 have discrete spectrums on M.

Assume that $\omega\in\Omega_B^1(\mathcal{F})$ is a basic harmonic 1-form of unit length. Then we define the operator $T_{\rm tr} : Q\otimes S(\mathcal{F})\to S(\mathcal{F})$ as

$$T_X\Psi = \nabla_X\Psi + \frac{1}{q-1}X\cdot D_{\rm tr}\Psi - \frac{1}{q-1}\omega(X)\omega\cdot D_{\rm tr}\Psi - \omega(X)\nabla_{\omega^\sharp}\Psi \tag{3.12}$$

for any $X \in \Gamma Q$ and $\Psi \in S(\mathcal{F})$. By a direct calculation, we have

$$|T_{\rm tr}\Psi|^2 = |\nabla_{\rm tr}\Psi|^2 - \frac{1}{q-1}|D_{\rm tr}\Psi|^2 - |\nabla_{\omega^\sharp}\Psi|^2 - \frac{1}{q-1}\mathrm{Re}\langle\kappa\cdot\Psi, D_{\rm tr}\Psi\rangle - \frac{2}{q-1}\mathrm{Re}\langle\nabla_{\omega^\sharp}\Psi, \omega\cdot D_{\rm tr}\Psi\rangle. \tag{3.13}$$

On the other hand, Lemma 3.3 implies

$$|D_{\rm tr}(\omega\cdot\Psi)|^2 = |\omega\cdot D_{\rm tr}\Psi|^2 + 4|\nabla_{\omega^\sharp}\Psi|^2 + 4\,\mathrm{Re}\langle\omega\cdot D_{\rm tr}\Psi, \nabla_{\omega^\sharp}\Psi\rangle. \tag{3.14}$$

Hence (3.13) and (3.14) yield

$$|T_{\rm tr}\Psi|^2 = |\nabla_{\rm tr}\Psi|^2 - \frac{1}{q-1}|D_{\rm tr}\Psi|^2 - \frac{q-3}{q-1}|\nabla_{\omega^\sharp}\Psi|^2 - \frac{1}{q-1}\mathrm{Re}\langle\kappa\cdot\Psi, D_{\rm tr}\Psi\rangle - \frac{1}{2(q-1)}\{|D_{\rm tr}(\omega\cdot\Psi)|^2 - |\omega\cdot D_{\rm tr}\Psi|^2\}.$$

By integrating the above equation with the Lichnerowicz type formula (3.10), we get

$$\int_M |T_{\rm tr}\Psi|^2 = \frac{q-2}{q-1}\int_M |D_{\rm tr}\Psi|^2 - \frac{1}{4}\int_M K^\sigma|\Psi|^2 - \frac{q-3}{q-1}\int_M|\nabla_{\omega^\sharp}\Psi|^2 - \frac{1}{2(q-1)}\int_M\{|D_{\rm tr}(\omega\cdot\Psi)|^2 - |\omega\cdot D_{\rm tr}\Psi|^2\} - \frac{1}{q-1}\int_M \mathrm{Re}\langle\kappa\cdot\Psi, D_{\rm tr}\Psi\rangle.$$

Let $D_b\Psi_1 = \lambda_1\Psi_1$, where λ_1 is the first eigenvalue of D_b. Since $\langle\Psi_1, X\cdot\Psi_1\rangle$ is pure imaginary, the last term in the above equation is zero. Hence we have

$$\int_M |T_{\rm tr}\Psi_1|^2 + \frac{q-3}{q-1}\int_M|\nabla_{\omega^\sharp}\Psi_1|^2 + \frac{1}{2(q-1)}\int_M\{|D_b(\omega\cdot\Psi_1)|^2 - |\omega\cdot D_b\Psi_1|^2\} = \int_M\left(\frac{q-2}{q-1}\lambda_1^2 - \frac{1}{4}K^\sigma\right)|\Psi_1|^2. \tag{3.15}$$

On the other hand, the Rayleigh inequality implies that for every spinor field Φ

$$\lambda_1^2 \le \frac{\int_M|D_b\Phi|^2}{\int_M|\Phi|^2}.$$

This means that for $\Phi = \omega\cdot\Psi_1$

$$\int_M |D_b(\omega\cdot\Psi_1)|^2 \ge \lambda_1^2\int_M|\omega\cdot\Psi_1|^2 = \int_M|\omega\cdot D_b\Psi_1|^2. \tag{3.16}$$

From (3.16), the left hand side of (3.15) is non-negative if $q \ge 3$. This implies the following theorem.

Theorem 3.4. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \ge 3$ and a bundle-like metric g_M such that $\Delta_B\kappa = 0$. Assume $K^\sigma > 0$. If M admits a non-trivial basic harmonic 1-form ω of constant length, then any eigenvalue λ of D_b satisfies*

$$\lambda^2 \ge \frac{q-1}{4(q-2)}\inf_M K^\sigma. \tag{3.17}$$

4. The limiting case

Now, we study the limiting case of (3.17). Let Ψ_1 be the eigenspinor corresponding to $\lambda_1 = \frac{q-1}{4(q-2)}\inf K^\sigma$ $(q \ge 3)$. From (3.15), Ψ_1 satisfies the following equation

$$T_{\rm tr}\Psi_1 = 0. \tag{4.1}$$

Equivalently, we have

$$\nabla_X \Psi_1 = -\frac{\lambda_1}{q-1} X \cdot \Psi_1 + \frac{\lambda_1}{q-1} \omega(X)\omega \cdot \Psi_1 + \omega(X)\nabla_{\omega^\sharp} \Psi_1. \tag{4.2}$$

We investigate the limiting case when $q > 3$ and when $q = 3$ because of (3.15).
Case (I) When $q > 3$. From (3.15), we get directly

$$\nabla_{\omega^\sharp} \Psi_1 = 0. \tag{4.3}$$

From (4.2) and (4.3), we have that for any $X \in Q$

$$\nabla_X \Psi_1 = -\frac{\lambda_1}{q-1} X \cdot \Psi_1 + \frac{\lambda_1}{q-1} \omega(X)\omega \cdot \Psi_1. \tag{4.4}$$

Taking Clifford multiplication of (4.4) with E_a yields

$$\kappa \cdot \Psi_1 = 0, \tag{4.5}$$

which implies $\kappa = 0$, i.e., $\mathcal{F}$ is minimal.

Now, we compute $\sum_a E_a \cdot R^S(X, E_a)\Psi_1$. From (4.4), we have that for any $X, Y \in \Gamma Q$

$$\left(\frac{q-1}{\lambda_1}\right)^2 \nabla_X \nabla_Y \Psi_1 = \{Y \cdot X - \omega(Y)\omega(X) - \omega(X) Y \cdot \omega - \omega(Y)\omega \cdot X\} \cdot \Psi_1 \\ - \frac{q-1}{\lambda_1}\{\nabla_X Y - \omega(\nabla_X Y)\omega - (\nabla_X \omega)(Y)\omega - \omega(Y)\nabla_X \omega\} \cdot \Psi_1. \tag{4.6}$$

Hence from (4.6) we have

$$\left(\frac{q-1}{\lambda_1}\right)^2 R^S(X, Y)\Psi_1 = \{(Y \cdot X - X \cdot Y) + \omega(X)(\omega \cdot Y - Y \cdot \omega) + \omega(Y)(X \cdot \omega - \omega \cdot X)\} \cdot \Psi_1 \\ + \frac{q-1}{\lambda_1}\{(\nabla_X \omega)(Y) - (\nabla_Y \omega)(X) + \omega(Y)\nabla_X \omega - \omega(X)\nabla_Y \omega\} \cdot \Psi_1. \tag{4.7}$$

Since ω is a basic harmonic form and $\kappa = 0$, we have $\sum_a E_a \cdot \nabla_{E_a}\omega = 0$. Hence (4.7) implies

$$\left(\frac{q-1}{\lambda_1}\right)^2 \sum_a E_a \cdot R^S(X, E_a)\Psi_1 \\ = 2q\{-X + \omega(X)\omega\} \cdot \Psi_1 + \frac{q-1}{\lambda_1}\left\{\nabla_X \omega - \sum_a (\nabla_{E_a}\omega)(X) E_a + \omega \cdot \nabla_X \omega\right\} \cdot \Psi_1. \tag{4.8}$$

Since ω has a unit length, $(\nabla_{E_a}\omega)(\omega^\sharp) = 0$. Hence (4.8) gives

$$(q-1)/\lambda_1 \sum_a E_a \cdot R^S(\omega^\sharp, E_a)\Psi_1 = \{\nabla_{\omega^\sharp}\omega + \omega \cdot \nabla_{\omega^\sharp}\omega\} \cdot \Psi_1. \tag{4.9}$$

On the other hand, we have that for any $Y \in \Gamma Q$

$$(d_B\omega)(\omega^\sharp, Y) = \omega^\sharp \omega(Y) - Y\omega(\omega^\sharp) - \omega([\omega^\sharp, Y]) \\ = (\nabla_{\omega^\sharp}\omega)(Y),$$

because $g_Q(\omega^\sharp, \nabla_Y \omega^\sharp) = 0$ for all Y. Hence $d_B\omega = 0$ yields

$$(\nabla_{\omega^\sharp}\omega)(Y) = 0 \quad \forall Y \in Q, \tag{4.10}$$

which means $\nabla_{\omega^\sharp}\omega = 0$. Therefore (3.8) and (4.9) imply

$$\rho^\nabla(\omega^\sharp) = 0. \tag{4.11}$$

Since ω is a basic harmonic 1-form and $\kappa = 0$, Corollary 2.2 and (4.8) imply that ω is parallel.
Case (II) When $q = 3$. By a direct calculation, (4.2) yields

$$\frac{1}{2}\kappa \cdot \Psi_1 = \omega \cdot \nabla_{\omega^\sharp} \Psi_1. \tag{4.12}$$

From (4.12), it is trivial that $\nabla_{\omega^\sharp} \Psi_1 = -\frac{1}{2}\omega \cdot \kappa \cdot \Psi_1$. Hence (4.2) gives that for any $X \in Q$

$$\nabla_X \Psi_1 = -\frac{1}{2}\{\lambda_1 X - \lambda_1 \omega(X)\omega - \omega(X)\omega \cdot \kappa\} \cdot \Psi_1. \tag{4.13}$$

Therefore (4.13) implies that for any $X, Y \in \Gamma Q$

$$\begin{aligned}\nabla_Y \nabla_X \Psi_1 = {} & \lambda_1^2/4\{X \cdot Y - \omega(Y) X \cdot \omega - \omega(X)\omega \cdot Y - \omega(X)\omega(Y)\} \cdot \Psi_1 \\ & - \lambda_1/2 \left\{\nabla_Y X - (\nabla_Y \omega)(X)\omega - \omega(\nabla_Y X)\omega - \omega(X)\nabla_Y \omega + \frac{1}{2}\omega(Y) X \cdot \omega \cdot \kappa \right. \\ & \left. + \frac{1}{2}\omega(X)\omega \cdot \kappa \cdot Y + \frac{1}{2}\omega(X)\omega(Y)\kappa - \frac{1}{2}\omega(X)\omega(Y)\omega \cdot \kappa \cdot \omega \right\} \cdot \Psi_1 \\ & + \frac{1}{2}\left\{(\nabla_Y \omega)(X)\omega \cdot \kappa + \omega(\nabla_Y X)\omega \cdot \kappa + \omega(X)\nabla_Y(\omega \cdot \kappa) + \frac{1}{2}\omega(X)\omega(Y)(\omega \cdot \kappa)^2\right\} \cdot \Psi_1. \end{aligned} \tag{4.14}$$

Since $\omega \cdot \kappa \cdot X - X \cdot \omega \cdot \kappa = 2\omega(X)\kappa - 2\kappa(X)\omega$ for $X \in \Gamma Q$, we have from (4.14)

$$\begin{aligned} R^S(Y, X)\Psi_1 = {} & \lambda_1^2/2\{X \cdot Y + g_Q(X, Y) - \omega(Y) X \cdot \omega + \omega(X) Y \cdot \omega\} \cdot \Psi_1 \\ & - \lambda_1/2\{(\nabla_X \omega)(Y)\omega - (\nabla_Y \omega)(X)\omega + \omega(Y)\nabla_X \omega - \omega(X)\nabla_Y \omega \\ & + \omega(Y)\kappa(X) - \omega(X)\kappa(Y)\} \cdot \Psi_1 \\ & + \frac{1}{2}\{\omega(X)\nabla_Y(\omega \cdot \kappa) - \omega(Y)\nabla_X(\omega \cdot \kappa) + (\nabla_Y \omega)(X)\omega \cdot \kappa - (\nabla_X \omega)(Y)\omega \cdot \kappa\} \cdot \Psi_1. \end{aligned} \tag{4.15}$$

Since ω is a basic harmonic 1-form, $\sum_a E_a \cdot \nabla_{E_a}\omega = -\mathrm{i}(\kappa^\sharp)\omega$. Hence (4.15) implies

$$\begin{aligned} 2\sum_a E_a \cdot R^S(Y, E_a)\Psi_1 = {} & \lambda^2\{\omega(Y)\omega - Y\}\Psi_1 \\ & - \lambda\left\{\omega(Y)\kappa \cdot \omega + \kappa(Y) + \sum_a (\nabla_{E_a}\omega)(Y) E_a \cdot \omega - \omega(Y)\omega(\kappa^\sharp)\right\} \cdot \Psi_1 \\ & - \left\{\nabla_Y \kappa + \omega(Y)\sum_a E_a \cdot \nabla_{E_a}(\omega \cdot \kappa) + \sum_a (\nabla_{E_a}\omega)(Y) E_a \cdot \omega \cdot \kappa\right\} \cdot \Psi_1. \end{aligned} \tag{4.16}$$

Since $\sum_a E_a \cdot \nabla_{E_a}(\omega \cdot \kappa) = -g_Q(\omega, \kappa)\kappa + |\kappa|^2\omega - 2\nabla_{\omega^\sharp}\kappa$, (4.16) with $Y = \omega^\sharp$ yields

$$2\sum_a E_a \cdot R^S(\omega^\sharp, E_a)\Psi_1 = \{-\lambda_1 \kappa \cdot \omega + \nabla_{\omega^\sharp}\kappa + g_Q(\omega, \kappa)\kappa - |\kappa|^2\omega\} \cdot \Psi_1. \tag{4.17}$$

From (3.8), we have

$$\langle\{\rho^\nabla(\omega^\sharp) + \nabla_{\omega^\sharp}\kappa + g_Q(\omega, \kappa)\kappa - |\kappa|^2\omega\} \cdot \Psi_1, \Psi_1\rangle = \lambda_1\langle\kappa \cdot \omega \cdot \Psi_1, \Psi_1\rangle. \tag{4.18}$$

Since $\langle X \cdot \Psi_1, \Psi_1\rangle$ is pure imaginary, the left hand side in (4.18) is pure imaginary. Hence the real part of the right hand side in (4.18) is also zero. That is,

$$\mathrm{Re}\langle\kappa \cdot \omega \cdot \Psi_1, \Psi_1\rangle = -g_Q(\omega, \kappa)|\Psi_1|^2 = 0, \tag{4.19}$$

which means $\omega(\kappa^\sharp) = g_Q(\omega, \kappa) = 0$. Hence from (4.13) we have

$$\nabla_{\kappa^\sharp}\Psi_1 = -\frac{1}{2}\lambda_1 \kappa \cdot \Psi_1. \tag{4.20}$$

From Lemma 3.3 and (4.20), we have

$$D_{\mathrm{tr}}(\kappa \cdot \Psi_1) = 0. \tag{4.21}$$

From the Lichnerowicz type formula (3.10), if $K^\sigma \geq 0$ and >0 at some point $x \in M$, then $\mathrm{Ker} D_{\mathrm{tr}} = \{0\}$. Hence (4.21) implies

$$\kappa \cdot \Psi_1 = 0, \quad \text{i.e.,}\ \kappa = 0. \tag{4.22}$$

Since $\kappa = 0$, (4.17) implies that $\rho^\nabla(\omega^\sharp) = 0$. From Corollary 2.2, ω is parallel.

Hence from the two cases, we have the following theorem.

Theorem 4.1. *Under the same condition as in Theorem 3.4, if there exists an eigenspinor field Ψ_1 of the basic Dirac operator D_b for the eigenvalue $\lambda_1^2 = \frac{q-1}{4(q-2)} \inf_M K^\sigma$, then $\mathcal{F}$ is minimal and ω is parallel.*

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References

[1] H. Baum, T. Friedrich, R. Grunewald, I. Kath, Twistor and Killing Spinors on Riemannian Manifolds, in: Seminarbericht Nr., vol. 108, Humboldt-Universität zu, Berlin, 1990.
[2] T. Friedrich, Dirac Operators in Riemannian Geometry, in: Graduate Studies in Mathematics, vol. 25, American Mathematical Society, Prov., 2000.
[3] J.F. Glazebrook, F.W. Kamber, Transversal Dirac families in Riemannian foliations, Comm. Math. Phys. 140 (1991) 217–240.
[4] G. Habib, Eigenvalues of the transversal Dirac operator on Kähler foliations, J. Geom. Phys. 56 (2005) 260–270.
[5] S.D. Jung, The first eigenvalue of the transversal Dirac operator, J. Geom. Phys. 39 (2001) 253–264.
[6] S.D. Jung, D.S. Kang, Eigenvalues of the basic Dirac operator and transversal twistor operator on Riemannian foliations, Far East J. Math. Sci. 14 (2004) 361–369.
[7] H.B. Lawson Jr., M.L. Michelsohn, Spin Geometry, Princeton Univ. Press, Princeton, New Jersey, 1989.
[8] P. March, M. Min-Oo, E.A. Ruh, Mean curvature of Riemannian foliations, Canad. Math. Bull. 39 (1996) 95–105.
[9] A. Mason, An application of stochastic flows to Riemannian foliations, Houston J. Math. 26 (2000) 481–515.
[10] A. Moroianu, L. Ornea, Eigenvalue estimates for the Dirac operator and harmonic 1-forms of constant length, C. R. Math. Acad. Sci. Paris 338 (2004) 561–564.
[11] Ph. Tondeur, Foliations on Riemannian Manifolds, Springer-Verlag, New-York, 1988.